Complex Calogero model with real energies

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Abstract

Recently, $\mathcal{PT}$ symmetry of many single-particle non-Hermitian Hamiltonians has been conjectured sufficient for keeping their spectrum real. We show that and how the similar concept of a "weakened Hermiticity" can be extended to some exactly solvable two- and three-particle models.

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1 Introduction

Insight into bound states in quantum mechanics is facilitated by solvable models. They clarify the structure of the single-particle wave functions in the context of supersymmetry [1], Lie algebras [2] and Sturm-Liouville oscillation theorems [3]. This approach can immediately be extended to the systems of more particles where a very exceptional role is played by the Calogero's exactly solvable Hamiltonian [4]

$$H^{(A)} = -\sum_{i=1}^{A} \frac{\partial^2}{\partial {x_i}^2} + \sum_{i<j=2}^{A} \left[\frac{1}{8}\,\omega^2\,(x_i - x_j)^2 + \frac{g}{(x_i - x_j)^2} \right] .$$

It describes A particles on a line with $2m = \hbar = 1$ and its Lie algebraic treatment proves enormously productive [5]. The analysis and interpretation of some of its properties is also facilitated in the non-singular limit $g \to 0$ where the interaction degenerates to the mere harmonic-oscillator attraction.

Recently, the so called $\mathcal{PT}$ symmetric quantum mechanics of Bender et al [6] offered a new picture of some single-particle models. For example, the most common spectrum of the harmonic oscillator in D dimensions was re-interpreted as a special case of a non-equidistant real spectrum pertaining to a slightly more general non-Hermitian model [7].

The picture is based on a complexification of coordinates which breaks the Hermiticity of the Hamiltonian but does not destroy the reality of the energies. In the present paper we intend to show that such a complexification method can be generalized and applied to some many-particle Hamiltonians. In a constructive way we are going to demonstrate the existence of non-Hermitian modifications of $H^{(2)}$ and $H^{(3)}$ which consequently preserve the reality of the spectrum.

2 Acceptable solutions

At any integer $A \geq 1$ the introduction of the centre of mass

$$R = R^{(A)} = \frac{1}{\sqrt{A}} \sum_{k=1}^{A} x_k$$

enables us to eliminate the bulk motion of the Calogero system. Submerging it for this purpose in an external auxiliary well $U^{(A)}(R^{(A)}) = A\,\omega^2[R^{(A)}]^2/8$ we get a slightly simplified version of the Calogero Hamiltonian,

$$\tilde{H}^{(A)} = H^{(A)} + U^{(A)} = \sum_{i=1}^{A} \left[-\frac{\partial^2}{\partial {x_i}^2} + \frac{A}{8}\,\omega^2\, x_i^2 \right] + \sum_{i<j=2}^{A} \frac{g}{(x_i - x_j)^2}\ .$$

This leads to the centre-of-mass equation

$$\left[-\frac{d^2}{dR^2} + \frac{A}{8}\,\omega^2 R^2 - \omega\,\sqrt{\frac{A}{8}}\,F \right] \Psi(R) = 0 \tag{1}$$

with the well known solutions [8]. Each element of its spectrum

$$F = F_N = 2N + 1, \qquad N = 0, 1, \ldots$$

adds a constant to all the internal energies E.

In the first nontrivial model $\tilde{H}^{(2)}$ with the mere two interacting particles let us ignore the centre-of-mass equation (1) as trivial and skip $g = 0$ (harmonic oscillator). Our Schrödinger bound state problem is then described by the singular ordinary differential equation in the relative coordinate $X = (x_1 - x_2)/\sqrt{2}$,

$$\left[-\frac{d^2}{dX^2} + \frac{1}{4}\,\omega^2 X^2 + \frac{1}{2}\,\frac{g}{X^2} - E \right] \psi(X) = 0. \tag{2}$$

The symbol E denotes the energy in the centre-of-mass system. The singularity resembles the current centrifugal term with a non-integer real parameter

$$\ell = \ell(g) = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{g}{2}}\ . \tag{3}$$

Such a singularity is, mathematically speaking, too strong. One has to reduce the domain to a half-axis and fix the ordering of the particles (say, $x_1 > x_2$, i.e., $X > 0$; sometimes, this is interpreted as a choice of the Bolzmann statistics).

In a mathematically more rigorous setting one should even demand that the repulsion is not weak, $g \geq 3/2$. Otherwise, the Hamiltonian admits many self-adjoint extensions, each of which may lead to a different spectrum [9, 10].

In the Calogero's paper [4] this problem has been addressed and resolved in a pragmatic spirit. Even when the irregular solution $\psi^{(-)}(X) \sim |X|^{-\ell}$, $|X| \ll 1$

becomes normalizable (which certainly happens for all $\ell < 1/2$ and/or $g < 3/2$) we eliminate it as "physically inacceptable" via an *ad hoc* condition. The details of this argument may be found in the recent comment [10] where the eligible *ad hoc* conditions were listed as depending on the range of g,

$$\begin{array}{ll} \lim_{X\to 0}\,[X^{-1/2}\psi^{(Hermitian)}(X)] = 0, & g \in (-1/2, 0), \\ \lim_{X\to 0}\,\psi^{(Hermitian)}(X) = 0, & g \in (0, 3/2), \\ \lim_{X\to 0}\,\psi^{(Hermitian)}(X) = 0, & g \in [3/2, \infty). \end{array} \tag{4}$$

The constraint $g/2 = \ell(\ell+1) > -1/4$ in the first line is unavoidable and protects the system against a collapse [11], while only the third line is fully equivalent to the conventional requirement of normalizability [12].

In what follows we shall use the same philosophy. The validity of the appropriate conditions of the type (4) will be postulated as a conventional regularization of the singularities. In the spirit of ref. [4] this will define the "acceptable" solutions and/or make them unique.

In the well known $A = 2$ case this implies the termination of the confluent hypergeometric series to the Laguerre polynomials,

$$\psi_n^{(Hermitian)}(X) \sim X^{\ell+1} \exp\left(-\frac{1}{4}\,\omega\,X^2\right) L_n^{\ell+1/2}\left(\frac{1}{2}\,\omega X^2\right), \qquad n = 0, 1, \ldots\ . \tag{5}$$

Although the two intervals $X > 0$ and $X < 0$ are impenetrably separated, one can deal with the presence of the singularity by another *physically motivated* postulate

$$\psi^{(bosonic)}(-X) = +\psi^{(bosonic)}(X), \qquad \psi^{(fermionic)}(-X) = -\psi^{(fermionic)}(X) \tag{6}$$

mimicking the Bose or Fermi statistics. Such a freedom is rendered possible by the full separation of the domains $X > 0$ and $X < 0$. By construction, the bosonic and fermionic wave functions vanish at the matching point $X = 0$. In what follows, a different type of the *ad hoc* symmetry will be employed and postulated. The resulting form of the modified statistics will be closely related to the so called $\mathcal{PT}$ symmetry in the single-particle quantum mechanics [6].

3 $\mathcal{PT}-$symmetric quantum mechanics

3.1 $A = 1$ and the complexification

No spikes exist in the trivial Hamiltonian $\tilde{H}^{(1)}$. It is equivalent to the harmonic oscillator but offers still a fairly nontrivial methodical lesson. In their pioneering letter [13] Bender and Boettcher emphasized that the complex shift of coordinates $R = R(r) = r - i\,\varepsilon,\ r \in (-\infty, \infty)$ in the harmonic oscillator Schrödinger equation (1) does not change its validity *and* preserves the normalizability of its wave functions $\Psi(R)$. The spectrum remains unchanged even when we admit a symmetric $r-$dependence in $\varepsilon = \varepsilon(r^2)$.

Empirically, the similar coexistence of the real spectrum with the non-Hermitian Hamiltonian has been detected for the various other single-particle potentials [14]. The phenomenon has attracted a lot of attention in the literature [15]. Its appealing interpretation has been conjectured in terms of the symbols $\mathcal{P}$ (which denotes parity, $\mathcal{P}R\mathcal{P} = -R$) and $\mathcal{T}$ (this is "time reversal" or complex conjugation, $\mathcal{T}i\mathcal{T} = -i$). For many Hamiltonians which commute with the product $\mathcal{PT}$, people have observed the reality of the spectra and attributed it *expressis verbis* to the $\mathcal{PT}$ symmetry [13, 16, 17].

In the present paper we intend to extend this language to cover also some systems of more particles.

3.2 $A = 2$ and the regularization

The complex shift of coordinates did not change the solutions of $A = 1$ equation (1) in the asymptotic region. The same is true for wave functions of the two Calogero particles. In the $A = 2$ equation (2) the formula

$$\frac{1}{X^2} = \frac{1}{x^2 + \varepsilon^2} - \frac{2\varepsilon^2}{(x^2 + \varepsilon^2)^2} + \frac{2ix\varepsilon}{(x^2 + \varepsilon^2)^2}, \qquad X = x - i\,\varepsilon(x^2)$$

indicates how the deformation of the integration contour regularizes the singularity. An immediate practical compensation of the loss of the Hermiticity of $\tilde{H}^{(2)}$ is found in an improvement of its regularity. This eliminates the rude constraint (4) and

opens space for new solutions. At the same time, we should not get too many of them [18] and so we demand the $\mathcal{PT}$ symmetry of the contour,

$$\mathcal{PT} \cdot X(x) \cdot \mathcal{PT} = X(-x). \tag{7}$$

It is possible to search for the even-parity-like solutions of the complex, regularized radial equation (2) in a way proposed originally by Buslaev and Grecchi [16] and dictated by the angular-momentum interpretation of the parameter ℓ [19]. In paper [7] we succeeded in complementing the known $\mathcal{PT}$-symmetrized solutions (5) of eq. (2) by the second hierarchy,

$$\psi_n^{(new)}(X) \sim X^{-\ell} \exp\left(-\frac{1}{4}\,\omega\, X^2\right) L_n^{-\ell-1/2}\left(\frac{1}{2}\,\omega X^2\right), \qquad n = 0, 1, \ldots . \tag{8}$$

At almost all values of g the pair of solutions (5) and (8) re-connects the subdomains $x > 0$ and $x < 0$. The equidistance of the old Hermitian spectrum is manifestly broken by the new even-like energies. Figure 1 illustrates the result. It displays both the energies

$$E_n^{(Hermitian)} = E^{(+)} = \frac{1}{2}\,\omega\,(4n + 2\ell + 3), \qquad E_n^{(new)} = E^{(-)} = \frac{1}{2}\,\omega\,(4n - 2\ell + 1)$$

(with $n = 0, 1, \ldots$) as functions of the coupling g (or rather $\ell = \ell(g)$) at a fixed value of the spring constant $\omega = 2 \cdot \sqrt{2/A}$.

We may summarize that at $A = 2$ the imaginary shifts of the single-particle coordinates x_1 and x_2 complexify the Jacobi coordinates R and X. This has comparatively trivial consequences for the regular centre of mass problem (1). The change proves much more influential in the singular radial equation (2). In the new language we were able to postulate the two types of behaviour near $X = 0$. *Vice versa*, the current return to the Hermitian constraint (4) acquires now an unconventional understanding of the reduction of the spectrum, caused by the pragmatic *ad hoc* elimination of all the redundant states (8).

4 Generalized statistics

4.1 $A = 2$ and the idea

The consistent complexification of our two-body singular equation (2) requires an introduction of an upward cut in the complex plane of X. This reflects the branching role of the singularity and implies that we have to choose $\varepsilon(0) = -Im\, X(0) > 0$. Still, having the sub-intervals $x > 0$ and $x < 0$ inter-connected, the freedom (6) of the choice of statistics seems lost. In fact, it is not. This is to be shown below.

The analysis is facilitated by the symmetry (7). We can return to Figure 1 and notice that the levels cross at $g = 2\,k^2 - 1/2$ for $k = 0, 1, 2, \ldots$. Everywhere off these exceptional points we can start from the $g = 0$ or $\ell = 0$ bound states with the well defined values of parity $= \pm 1$. We propose to use a smooth continuation in g. This transfers the label ± 1 to almost all the energies. We re-introduce the complexified bosonic and fermionic symmetry in one of the most natural ways.

In a less intuitive setting we have to imagine that the parity is not conserved. At $A = 2$ we are able to replace this concept by the equivalent permutation symmetry. Our knowledge of the explicit wave functions (5) and (8) enables us to speak about the bosons and fermions defined by the following rule

$$\begin{aligned} \psi^{(bosonic)}(-X) &= (-1)^{-\ell}\psi^{(bosonic)}(X), \\ \psi^{(fermionic)}(-X) &= (-1)^{\ell+1}\psi^{(fermionic)}(X). \end{aligned} \tag{9}$$

This is a natural generalization of the $\ell = 0$ *ad hoc* conditions (6) prescribing the behaviour near the singularity. Mathematically, it enables us to discretize the bound state spectrum in a way employed also in refs. [7, 16, 20].

Whenever we choose an integer ℓ the role of the parity becomes partially re-established. The new definition of the statistics (9) preserves many features of its Hermitian predecessor (6) also in the limit $\lim_{x\to\infty} \varepsilon(x^2) = 0$. The number of nodal zeros can be shared by our new bosons and fermions. Their spectra are mutually shifted. Geometrically they are formed by the opposite branches of the energy parabolas in Figure 1. In terms of an abbreviation $\alpha = \ell + 1/2$ the transition from fermions to bosons is a mere change of the sign at this square root, $\alpha = \sqrt{1/4 + g/2} \to -\alpha = -\sqrt{1/4 + g/2}$.

4.2 $A = 3$ and a toy model

The solution of the Calogero three-body problem proves significantly facilitated after one omits two of its spikes. For the purely Hermitian toy Hamiltonian

$$\hat{H}^{(T)} = \sum_{i=1}^{3} \left[-\frac{\partial^2}{\partial {x_i}^2} + \frac{3}{8}\,\omega^2\, x_i^2 \right] + \frac{g}{(x_1 - x_2)^2}$$

this has been noted in the first half of Section 3 in ref. [4]. After a constant imaginary shift of the variables x_1, x_2 and x_3 in the three-body Hamiltonian $\hat{H}^{(T)}$ the centre of mass $R = (x_1 + x_2 + x_3)/\sqrt{3}$ becomes complex. Its elimination is still trivial. The other two Jacobi coordinates

$$X = \frac{x_1 - x_2}{\sqrt{2}}, \qquad Y = \frac{x_1 + x_2 - 2\,x_3}{\sqrt{6}}$$

enter the partial differential Schrödinger equation

$$\left[-\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} + \frac{3}{8}\,\omega^2 (X^2 + Y^2) + \frac{1}{2} g_3 X^{-2} - E \right] \Phi(X, Y) = 0.$$

At $A = 3$ it replaces the $A = 2$ radial equation (2). We employ its separability in the hyperspherical coordinates ρ and ϕ. Under the guidance of the Calogeros's paper [4] this gives $X = \rho\, \sin\phi$ and $Y = \rho\, \cos\phi$ while $\Phi(X, Y) = \psi(\rho) f(\phi)$. The minor relevance of the centre-of-mass motion extends naturally to the $\rho-$dependence. With $\rho \in (0, \infty)$ it is described by the wave functions

$$\psi_{n,k}(\rho) = \rho^{\beta(k)} \exp\left(-\sqrt{\frac{3}{32}}\omega\, \rho^2 \right)\, L_n^{\beta(k)} \left(\sqrt{\frac{3}{8}}\omega\, \rho^2 \right), \qquad n, k = 0, 1, \ldots$$

with the energies

$$E = E_{n,k} = \sqrt{\frac{3}{2}}\,\omega\, [2n + 1 + \beta(k)], \qquad n, k = 0, 1, \ldots\ .$$

Both depend on g in a way mediated by the new quantity $\beta(k) > 0$. It is defined as the square root of the eigenvalue of the "innermost" hyperangular equation

$$\left(-\frac{d^2}{d\phi^2} + \frac{g}{2\, \sin^2 \phi} \right) \chi_k(\phi) = \beta^2(k)\, \chi_k(\phi). \tag{10}$$

Our key idea is to complexify solely the third coordinate ϕ. We shall use the recipe

$$\phi = \xi - i\, \varepsilon(\xi), \qquad \xi \in (-\pi, \pi)$$

inspired by its $A = 2$ predecessor. The "Bose" or "Fermi" statistics of our toy solutions will be determined by their regularized behaviour (9) with $X(x)$ replaced by $\phi(\xi)$. In effect the complex double well problem (10) is appropriately constrained in a way consistent with the permutations of x_1 and x_2, i.e., with the $\mathcal{PT}$-like symmetry

$$\mathcal{PT} \cdot \phi(\xi) \cdot \mathcal{PT} = \phi(-\xi)$$

and with the ordinary real parity conservation reflecting the unconstrained variability of x_3,

$$\chi_k(\pi - \phi) = (-1)^k \, \chi_k(\phi) \ . \tag{11}$$

This establishes the closest parallels between $A = 2$ and $A = 3$.

5 Spectra

5.1 Explicit solutions of the toy model

Differential equation (10) possesses a general solution which is a hypergeometric series of the Gauss type,

$$\chi^{(\pm)}(\phi) = (\sin\,\phi)^{1/2\pm\alpha}\ {}_2F_1(u^{(\pm)}, v^{(\pm)}; 1 \pm \alpha; \sin^2\phi), \quad \alpha = \frac{1}{2}\sqrt{1+2g} > 0$$

with $2u^{(\pm)} = 1/2 - \beta \pm \alpha$ and $2v^{(\pm)} = 1/2 + \beta \pm \alpha$ and with an equivalent alternative form

$$\chi^{(\pm)}(\phi) = \cos\,\phi\,(\sin\,\phi)^{1/2\pm\alpha}\ {}_2F_1(u^{(\pm)} + 1/2, v^{(\pm)} + 1/2; 1 \pm \alpha; \sin^2\phi).$$

Only the regular, $^{(+)}$−superscripted states were acceptable in ref. [4]. For us, the "irregular" $\chi^{(-)}(\phi)$ represent bosonic states. These functions are smooth and bounded due to our complex regularization of eq. (10). We can derive their estimate $\chi^{(\pm)}(\phi) \sim \varepsilon^{1/2\pm\alpha}[1 + \mathcal{O}(\varepsilon^2)]$ for the non-vanishing $\varepsilon \approx \varepsilon(0) > 0$ and in the closest vicinity of the singularities $\xi = 0$ and $\xi = \pm\pi$, i.e., for $\xi \in (-\varepsilon, \varepsilon)$, $\xi \in (-\pi, -\pi + \varepsilon)$ or $\xi \in (\pi - \varepsilon, \pi)$. This implies that both the signs can equally well appear at the parameter α.

On every boundary of convergence $\sin^2\phi = 1$ of our power-series solutions we arrive at the same necessity of termination as before. In the light of the "real" symmetry (11) this occurs if an only if our free parameter belongs to the sequence

$$\beta(k) = \beta^{(\pm)}(k) = k \pm \alpha + 1/2, \qquad k = 0, 1, \ldots \ .$$

In a way paralleling the Calogero's construction this leads to the Gegenbauer polynomials in

$$\chi_k^{(fermions/bosons)}(\phi) = (\sin\,\phi)^{1/2\pm\alpha}\ C_k^{1/2\pm\alpha}(\cos\,\phi), \qquad \alpha = \frac{1}{2}\sqrt{1+2g} > 0.$$

An analogy of this key formula with its $A = 2$ predecessor becomes clearer in the older notation with $1/2 + \alpha = \ell + 1$ and $1/2 - \alpha = -\ell$.

The set of the toy eigenvalues β^2 decays in the two subsets. They differ just by the sign attached to the square-root parameter α. The energy spectrum emerges in the compact form

$$E_{n,k}^{(\pm)} = \sqrt{\frac{3}{8}}\omega\,(4n + 2k \pm 2\alpha + 3), \qquad \alpha = \frac{1}{2}\sqrt{1+2g} > 0, \qquad n, k = 0, 1, \ldots .$$

In the convenient scale $\sqrt{\frac{3}{8}}\omega = 1$, it is sampled in Figure 2.

5.2 The genuine three-body Calogero-type system

The resemblance between equations (2) and (10) is not too surprising since in our toy model the third particle is only bound by the purely harmonic forces. Let us now recall the full-fledged Calogero Hamiltonian

$$\tilde{H}^{(3)} = \sum_{i=1}^{3}\left[-\frac{\partial^2}{\partial {x_i}^2} + \frac{3}{8}\,\omega^2\,x_i^2\right] + \frac{g}{(x_1-x_2)^2} + \frac{g}{(x_2-x_3)^2} + \frac{g}{(x_3-x_1)^2}$$

and recollect that the related partial differential Schrödinger equation

$$\left[-\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} + \frac{3}{8}\,\omega^2(X^2+Y^2) + \frac{1}{2}g\,X^{-2}\right.$$

$$\left. + \frac{1}{2}g\,(\sqrt{3}Y - X)^{-2} + \frac{1}{2}g\,(\sqrt{3}Y + X)^{-2} - E\right]\,\Phi(X,Y) = 0$$

degenerates to the ordinary differential "innermost" equation

$$M\, f_k(\phi) = \beta^2(k) M\, f_k(\phi)$$

containing the hyperspherical momentum operator

$$M = -\frac{\partial^2}{\partial\phi^2} + \frac{1}{2}\left(\frac{g_3}{\sin^2\phi} + \frac{g_1}{\sin^2(\phi + \frac{2}{3}\pi)} + \frac{g_2}{\sin^2(\phi - \frac{2}{3}\pi)}\right).$$

For the equal couplings $g_j = g$ we have the complex differential equation

$$\left(-\frac{d^2}{d\phi^2} + \frac{9\, g}{2\, \sin^2 3\,\phi}\right)\chi_k(\phi) = \beta^2(k)\, \chi_k(\phi). \tag{12}$$

It is a straightforward six-well modification of the above toy double-well problem with $\xi \in (-\pi, \pi)$. In its analysis let us start from the Hermitian limit $\varepsilon = 0$ with the clear geometrical meaning of the permutation symmetry in the real $X - Y$ plane. It is divided into six subdomains separated by the impenetrable two-body barriers. Thus, the $x_1 \leftrightarrow x_2$ interchange $\mathcal{P}_{(1-2)}$ happens precisely along the $Y-$axis. Similarly, the other two permutations $x_2 \leftrightarrow x_3$ and $x_3 \leftrightarrow x_1$ take place along the respective lines $Y = \pm X/\sqrt{3}$ [4].

As soon as we introduce $\varepsilon \neq 0$ the action of the complexified permutations has to be subject to the triple rule

$$\begin{aligned} \mathcal{PT}_{(1-2)} \cdot \phi(\xi) \cdot \mathcal{PT}_{(1-2)} &= \phi(-\xi), \\ \mathcal{PT}_{(2-3)} \cdot \phi(\xi) \cdot \mathcal{PT}_{(2-3)} &= \phi(\tfrac{2}{3}\pi - \xi), \\ \mathcal{PT}_{(3-1)} \cdot \phi(\xi) \cdot \mathcal{PT}_{(3-1)} &= \phi(-\tfrac{2}{3}\pi - \xi) \end{aligned} \tag{13}$$

which guarantees the commutativity of the Hamiltonian with the vectorial operator $\overrightarrow{\mathcal{PT}}$. In analogy with equation (9) we postulate that our solutions satisfy the six *ad hoc* boundary conditions

$$\begin{aligned} f^{(bosonic)}(-\phi_j) &= (-1)^{-\ell} f^{(bosonic)}(\phi_j), \\ f^{(fermionic)}(-\phi_j) &= (-1)^{\ell+1} f^{(fermionic)}(\phi_j), \\ \phi_j = \phi - \tfrac{j}{3}\pi, &\qquad j = -2, -1, 0, 1, 2, 3. \end{aligned} \tag{14}$$

This extends the $\mathcal{PT}-$symmetric quantum mechanics of ref. [6] to our complexified system of three particles.

5.3 Solutions

The parameter $\ell = \ell(g)$ is the same function (3) of g as above. Incidentally, it vanishes in the harmonic-oscillator limit $g \to 0$. Explicit solutions of our Calogero-inspired model can be constructed with an ample use of its toy predecessor. In particular, the toy hypergeometric solutions have only to be subject to the modified boundary conditions. This parallels the Hermitian situation.

Firstly, due to the overall symmetry (14) it is still natural to match the wave functions in the middle of the separate wells. Formally this relies on the fact that all of the infinite hypergeometric series reach their limits (i.e., radii) of convergence precisely at/along the three real-middle-of-the-well lines $Y = 0$ ($= X-$axis) and $Y = \pm\sqrt{3}X$. For this reason they have to degenerate to the Gegenbauer polynomials as before.

Secondly, the exact values of the energies of the triple-spike complex Calogero model follow immediately from a mere comparison of equations (10) and (12). The appropriate re-scaling of the "intermediate" eigenvalue parameter β gives the resulting final form of the spectrum

$$E^{(\pm)}_{n,k} = \sqrt{\frac{3}{8}}\omega\,(4n + 6k \pm 6\alpha + 5), \qquad \alpha = \frac{1}{2}\sqrt{1+2g} > 0, \qquad n, k = 0, 1, \dots\ .$$

It is illustrated in Figure 3. The pertaining complex angular wave functions

$$f_k^{(fermionic)}(\phi) = \chi_k^{(fermions)}(3\phi), \qquad f_k^{(bosonic)}(\phi) = \chi_k^{(bosons)}(3\phi),$$

and their oscillator-like radial counterparts

$$\psi^{(\pm)}_{n,k}(\rho) = \rho^{3k\pm3\alpha+3/2} \exp\left(-\sqrt{\frac{3}{32}}\omega\,\rho^2\right)\ L_n^{3k\pm3\alpha+3/2}\left(\sqrt{\frac{3}{8}}\omega\,\rho^2\right)$$

are easily determined using the elementary insertions in the obvious manner resembling the classical Calogero's analysis.

6 Summary

In a way guided by the available one-body experience we complexified the two- and three-body Calogero model, having consequently employed the advantage of its

separability. This enabled us to suppress the majority of complications (e.g., an interplay between several complex variables) which would necessarily arise in any more complicated many-body model.

Our conclusion of upmost importance concerns the new role of the operator $\mathcal{PT}$. We were guided by its $A = 2$ action upon $X \sim x_1 - x_2$ as given by equation (7). This equation was interpreted as a complexification of the usual permutation of particles $x_1 \leftrightarrow x_2$. Hence, we also assigned a new meaning to the whole concept of the $\mathcal{PT}$ symmetry for more particles. Its re-examination in the three-body context enabled us to find its appropriate permutation-symmetry generalization (13).

During our application of the complex permutation symmetry to the Calogero model we arrived at several satisfactory analogies between its separate $A = 1$, $A = 2$, and $A = 3$ special cases. Firstly, as long as we studied the complex coordinates only on the "innermost" level, we were able to leave all the previous stages (with any complex Jacobi coordinates) open to virtually arbitrary analytic continuation.

Secondly, it was comparatively easy to specify the correct physical states in the limit of the standard Calogero system. This backward relationship to Hermitian predecessors is quite instructive. Among our bosonic as well as fermionic complex solutions some of them "survive" in the limit $\varepsilon \to 0$. Still, as long as the domains themselves become halved, the full set of solutions is overcomplete. For this reason the $\mathcal{PT}$ symmetric bosons were declared to be redundant. They must be eliminated by brute force. Usually, for physical reasons, their function in statistics is successfully mimicked by their non-analytic Calogero substitutes (6).

We may summarize that the reality of spectra of our new, non-Hermitian and complexified Calogero model can be again tentatively interpreted as stemming from its multiple permutation $\overrightarrow{\mathcal{PT}}$ symmetry. We can conjecture, therefore, that this type of a complex permutation could presumably play the role of a certain "weakened Hermiticity" in the many-body physics. This generalizes the parity $\times$ time-reversal symmetry which is already comparatively well understood within the $\mathcal{PT}$ symmetric quantum mechanics of isolated particles [6].

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Figure captions

Figure 1. $\mathcal{PT}$ symmetric energies for $A = 2$

Figure 2. Toy spectrum

Figure 3. $\mathcal{PT}$ symmetric energies for $A = 3$

Figure 1. PT symmetric energies for A = 2

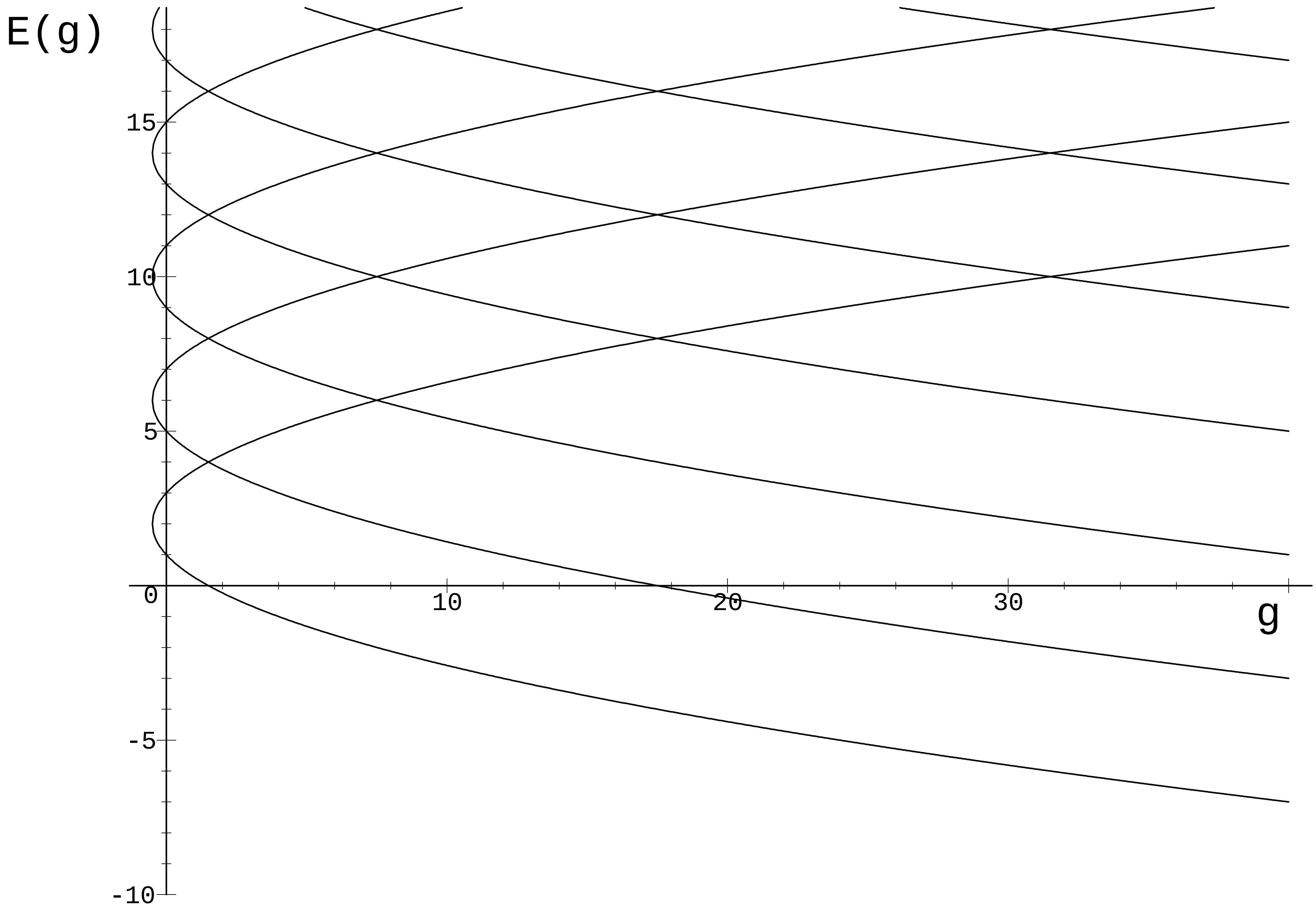

Figure 2. Toy spectrum

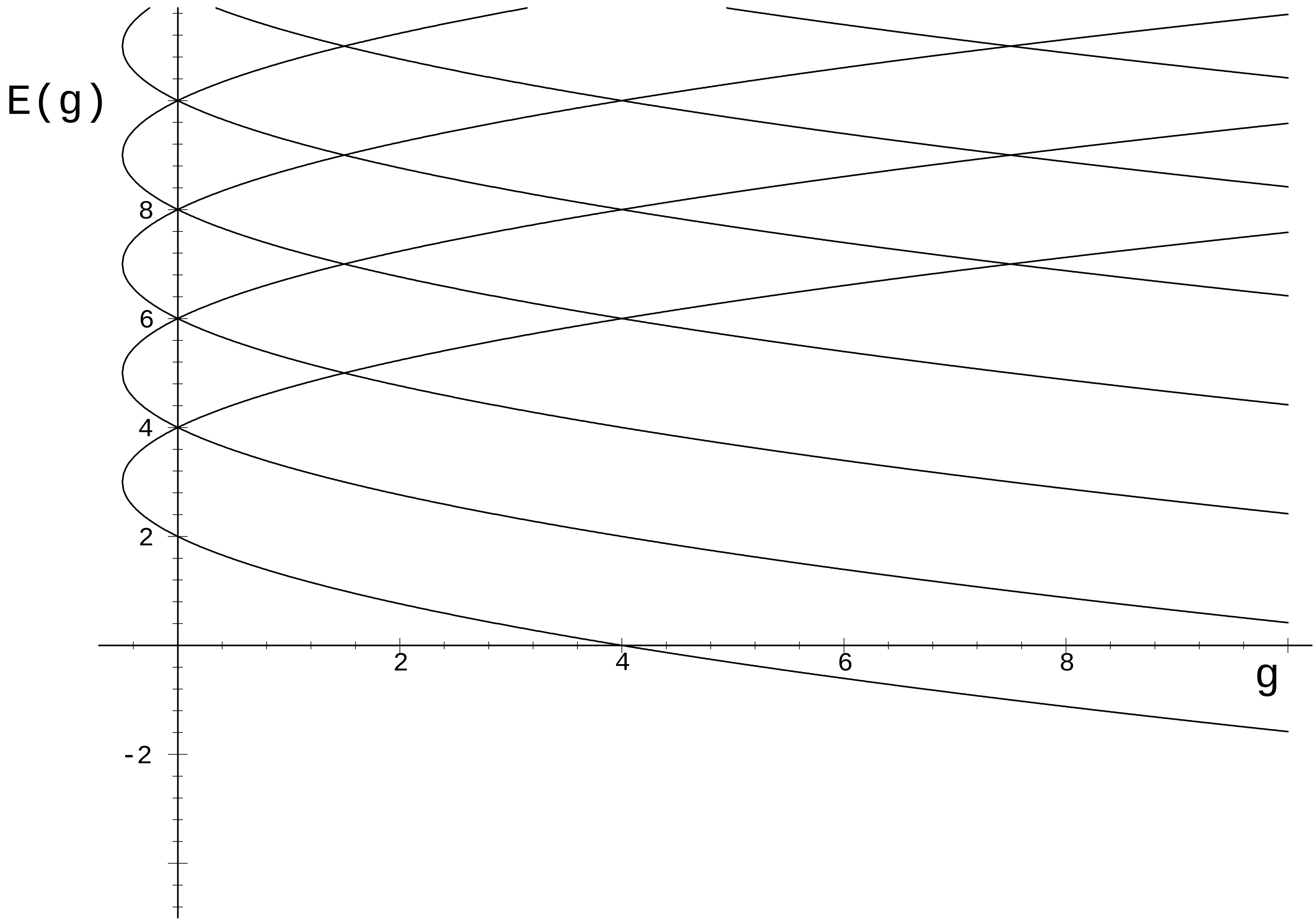

Figure 3. PT symmetric energies for A = 3